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Manuscript received February 14, 1966; revision received September 23, 1966; paper accepted September 23, 1966. Paper presented at A.I.Ch.E. Dallas meeting.

Model-Reference Adaptive Control System

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An adaptive process control scheme—which uses a differential equation model, requires no differentiations in the adaptive circuitry, and no identification of the varying process parameter —was analyzed mathematically and studied on an analog computer. The adaptive loop is operative only during a transient and corrects only in the direction of mismatch between the process and model. Linear analysis of the system differential equations when the correction signal is not introduced into the controller provides a means of designing the adaptive circuitry and approximating the effect when the signal is introduced. The effects on system stability with a pure delay, measurement lag, or an additional pole in the process are presented. Computer results show that when large differences exist between the actual process time constant and the original controller setting, the adaptive feature reduces the overshoot to essentially zero.

Chemical engineers have been slow in applying quantitative principles of feedback control theory to the design of process control systems. The major factor limiting the application of this theory to processing systems is the lack of information about process dynamics. This situation has resulted in a serious curtailment of the full use of available process control capability. The majority of the installations utilize the field-adjustable, three-mode controllers in the control scheme that duplicates the control functions when the process is operated manually. Thus, prior experience is the primary design basis in the process industry rather than a quantitative application of control theory.

In recent years, there has been increasing interest and research in process dynamics as well as a major increase in the number of engineers with good background in control theory. This situation offers the possibility of applying more sophisticated control schemes to effect superior control performance in critical applications.

One such possibility is the use of adaptive control schemes. For the work presented here an adaptive process control system is defined as a system which can alter its response to changes in the process to be controlled and that this be accomplished by measurements of input and/or response and the corresponding automatic adjustment of one or more controller parameters. One of the most difficult problems when designing an adaptive control system is the determination of process parameters, that is, process identification. Determination of the gain and time constant of a first-order system is substantially complex and the difficulty increases with process order. A number of adaptive systems dependent primarily upon system identification have appeared in the recent literature (*1* to *5*).

One very important approach to the problem of system identification is the general model-reference [MR] scheme. Oppeldahl (*6*) devised and analyzed a MR system applicable to any order system, but this method requires first- and higher order derivatives. Marcus and Hougen (*7*) have applied the MR technique to the adaptive control of a heat exchanger with slowly varying time constant. The circuitry used requires taking the derivative of the model and process outputs. Nevertheless, the results obtained by analog simulation appear promising. There is no question that the identification problem can be solved when no restriction is placed on the number of derivatives taken. When the adaptive and compensation networks are analog computers, this approach is questionable from a practical standpoint. The MR technique nonetheless appears to offer many possibilities for a number of process control problems.

The method presented here uses a differential equation model, requires no differentiation and no identification of the varying parameter. The adaptive loop is operative only during a transient and corrects only in the direction of mismatch between the system and model. Adaptation is for a single varying parameter, although other param-

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eters may vary somewhat without affecting the adaptive loop critically.

SYSTEM EQUATIONS: DERIVATION AND ANALYSIS

General Equations

Consider a first-order process with the transfer function

$$G_p(s) = \frac{T_o(s)}{P(s)} = \frac{K_p}{\tau s + 1} \quad (1)$$

where τ varies slowly and randomly with time (but is constant during the period of a transient) and a controller of the form.

$$G_c(s) = \frac{P(s)}{E(s)} = \frac{K_c(\tau' s + 1)}{s} \quad (2)$$

For the simple feedback control of this process, the overall transfer function is given by

$$G_o(s) = \frac{T_o(s)}{T_r(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{K_pK_c(\tau' s + 1)}{s(\tau s + 1) + K_pK_c(\tau' s + 1)} \quad (3)$$

With $K_pK_c = K$

$$G_o(s) = \frac{K(\tau' s + 1)}{s^2 + (1 + K\tau')\,s + K} \quad (4)$$

A differential equation reference model which has the desired response is constructed with $\tau = \tau'$. If $T_{om}(s)$ is the model output, then its transfer function becomes

$$G_m(s) = \frac{T_{om}(s)}{T_r(s)} = \frac{K}{s + K} \quad (5)$$

For the adaptive system in Figure 1 we have

$$\text{Response error} = F(s) = T_{om}(s) - T_o(s) \quad (6)$$

$$\text{Correction signal} = L(s) = B(s)F(s) \quad (7)$$

Combining Equations (4) to (7) and letting $(\tau - \tau') = \Delta\tau$, we obtain

$$\frac{F(s)}{T_r(s)} = \frac{Ks^2\,\Delta\tau}{(s + K)\,[\tau s^2 + (1 + K\tau')\,s + K]} \quad (8)$$

$$\frac{L(s)}{T_r(s)} = \frac{B(s)Ks^2\,\Delta\tau}{(s + K)\,[\tau s^2 + (1 + K\tau')\,s + K]} \quad (9)$$

An important consideration is the form of the correction signal $l(t)$, when it is not introduced into the controlled process; that is, the closed loop $G_c(s)G_p(s)B(s)$ is broken between $B(s)$ and $G_c(s)$. This leads to a choice of $B(s) = B$ or $B(s) = B/s$ to ensure a steady state correction signal of zero or a constant.

Writing Equation (4) as a differential equation, we get

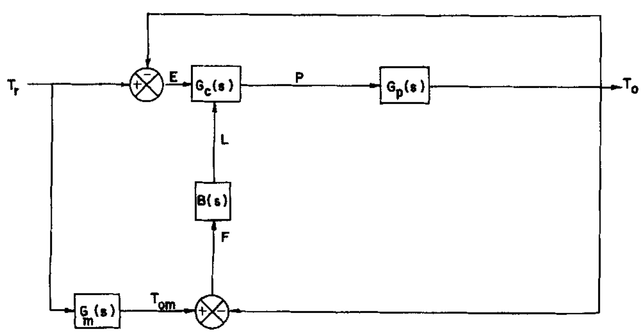


Fig. 1. Adaptive system.

$$\tau\frac{d^2T_o}{dt^2} + (1 + K\tau')\frac{dT_o}{dt} + KT_o = K\tau'\frac{dT_r}{dt} + KT_r$$

since $\tau' = l(t) + \tau'$ initial $= l(t) + \tau''$, we have

$$\ddot{T}_o + \left[\frac{1 + K\,(l(t) + \tau'')}{\tau}\right]\dot{T}_o + \frac{KT_o}{\tau} = \frac{K}{\tau}\left[(l(t) + \tau'')\,\dot{T}_r + T_r\right] \quad (10)$$

Similarly for $B(s) = B/s$, Equation (9) becomes

$$\dddot{l} + \left[\frac{1 + K\,(\tau + l(t))}{\tau}\right]\ddot{l} + \left[\frac{l + K\,(1 + l(t))}{\tau}\right]\dot{l} + K^2\,l = KB\,(\tau - l(t))\dot{T}_r \quad (11)$$

Equations (10) and (11) are nonlinear with the general forms

$$\ddot{T}_o + a(T)\dot{T}_o + bT_o = c(T)\dot{T}_r + bT_r \quad (12)$$

$$\dddot{l} + f(l)\,\ddot{l} + g(l)\,\dot{l} + hl = m(l)\,\dot{T}_r \quad (13)$$

Stability Analysis

The stability of the solutions of Equations (12) and (13) can be examined and the parameters affecting the rate of convergence determined by application of the direct method of Lyapunov (*8*). Lyapunov's theorem states that a system is stable if a scalar function $V(x_1, x_2, \ldots x_n)$ is found with the following properties:

1. Outside the origin, $V(x_1, x_2, \ldots x_n) > 0$
2. $V(0) = 0$
3. $V(x_1, x_2, \ldots x_n)$ is continuous and has continuous first partial derivatives in a region R about the origin (*13*).
4. $\dot{V} = \dfrac{\partial V(x_1, x_2, \ldots x_n)}{\partial t} \leqq 0$ in R

Considering first Equation (10), we have for the free motion

$$\ddot{T}_o + \left[\frac{1 + K\,(l(t) + \tau'')}{\tau}\right]\dot{T}_o + \frac{K}{\tau}T_o = 0$$

since $l(t) = B\int_0^t (T_{om} - T_o)\,dt = g(T_o, T_m, t)$ and letting $K/\tau = b$ we have

$$\ddot{T}_o + F(T_o, T_m)\,\dot{T}_o + bT_o = 0 \quad (14)$$

If $y_1 = T_o$ and $y_2 = \dot{y}_1$, the canonical form is

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -by_1 - F(y_1, T_m)\,y_2$$

The method outlined by Ingwerson (*11*) is used to obtain a Lyapunov function of the form

$$V = \frac{b}{2}y_1^2 + \frac{y_2^2}{2}$$

then

$$\dot{V} = y_2\frac{\partial V}{\partial y_1} - [by_1 + F(y_1, T_m)y_2]\frac{\partial V}{\partial y_2}$$

$$\dot{V} = -F(y_1, T_m)y_2^2 \quad (15)$$

Conditions 1 to 4 are satisfied if $b > 0$ and $F(y_1, T_m) > 0$.

For a nonlinear differential equation with the form

$$\dddot{x} + f(x, \dot{x}, \ddot{x})\,\ddot{x} + g(x, \dot{x})\,\dot{x} + h(x)\,x = 0 \quad (16)$$

Goldwyn (*12*) has determined the conditions for asymptotic stability.

Since Equation (13) is a special case of Equation (16) the stability conditions are

$$h, f(l), g(l) > 0 \text{ for } l \neq 0$$
$$f(l)g(l) - h > 0$$

With these criteria established it is now possible to determine under what conditions $l_{ss} \rightarrow 0$. If

$$\dddot{l} + f(l))\ddot{l} + g(l)\dot{l} + hl = 0$$

then, provided that $f(l)$, $g(l)$, and $h \neq 0$ at any time t, $\dddot{l} = \ddot{l} = \dot{l} = l = 0$. Obviously a condition which satisfies these requirements is $(dT_r/dt) = 0$, or $T_r =$ constant, since Equation (11) reduces to Equation (13) and $f(l), g(l)$, and $h \neq 0$.

The rate of convergence of the Lyapunov function can be examined in the following manner. Starting with Equations (17) and (18)

$$V = \frac{b}{2} y_1^2 + \frac{y_2^2}{2} \tag{17}$$

$$\dot{V} = -bG(y_1, T_m)y_1 \tag{18}$$

where

$$G(y_1, T_m)y_1 = \int F(y_1, T_m)\, dy_1$$
$$= \int_o^{y_1} \int_o^t B(T_m - y_1)\, dt dy_1$$

assume the solution of Equation (17) to be of the form

$$V(t) = V(0)e^{f(b, y_1, G)} \tag{19}$$

Then

$$\ln \frac{V(t)}{V(0)} = f(b, y_1, G)$$

Taking the derivative with respect to time

$$\frac{\dot{V}(t)}{V(0)} = \frac{d f(b, y_1, G)}{dt}$$

and

$$\int_o^t \frac{\dot{V}(t)}{V(t)}\, dt = f(b, y_1, G) \tag{20}$$

substituting Equations (17) and (18) into (20), we get

$$f(b, y_1, G) = -2 \int_o^t \left[\frac{by_1G}{by_1^2 + y_1^2} \right] dt \tag{21}$$

Finally, combination of Equations (19) and (21) yields

$$V(t) = V(0) \exp \left[-2 \int_o^t \left[\frac{by_1G}{by_1^2 + y_2^2} \right] dt \right]$$

$$V(t) = V(0) \exp \left[-2 \int_o^{y_1} \int_o^t \int_o^t \left[\frac{by_1B(T_m - y_1)}{by_1^2 + y_2^2} \right] dt^2\, dy_1 \right] \tag{22}$$

$V(t)$ is a monotonically decreasing function and the rate of convergence may be increased by increasing the controllable parameter B.

Start-Up of Chemical Reactor

Consider the stirred-tank chemical reactor. A first-order reaction, given by $A \xrightarrow{k} B$, where K is the specific rate constant, is occurring.

Making a material balance on the system, assuming a perfectly mixed vessel, applying an energy balance, and assuming the case of an endothermic reaction, we get

$$\frac{dT}{dt} + \left[\frac{UA + v_{\rho o} C_{po}}{\rho V_T C_p} \right] T = \frac{UA}{\rho V_T C_p} T_j + \frac{Qk}{\rho C_p} C_a + \frac{v_{\rho i} C_{pi}}{\rho V_T C_p} T_i \tag{23}$$

and in transformed notation

$$\left[s + \left[\frac{UA + v_{\rho o} C_{po}}{\rho V_T C_p} \right] \right] T(s) = \frac{UA}{\rho V_T C_p} T_j(s) + \frac{Qk}{\rho C_p} C_a(s) + \frac{v_{\rho i} C_{pi}}{\rho V_T C_p} T_i(s) \tag{24}$$

For the temperature control of the reactor a controller and valve are required. The controller output P for a proportional and integral controller is given by

$$P(t) = K_c(\tau' + \int edt)$$

or

$$\frac{P(s)}{E(s)} = \frac{K_c(\tau' s + 1)}{s} \tag{25}$$

and the valve transfer function for a linear valve with small inertia is

$$\frac{V(s)}{P(s)} = K_v \tag{26}$$

If we assume $\rho = \rho_i = \rho_o$ and $C_p = C_{po}$, the transfer function of the process becomes

$$\frac{T_o(s)}{T_j(s)} = \frac{K_p}{\tau s + 1} \tag{27}$$

where

$$K_p = \frac{UA}{UA + v_\rho C_p} \text{ and } \tau = \frac{\rho V_T C_p}{UA + v_\rho C_p}$$

Consider the start-up of the batch process, where the temperature must be rapidly increased from 20° to 100°C. and then held constant at 100 ± 0.5°C. to complete the reaction and any overshoot is considered undesirable. If a controller with proportional action alone were used, then the steady state error is given by

$$e_{ss} = \lim_{s \to o} \left[\frac{sT_r(s)}{1 + G(s)} \right]$$

since

$$T_r = \frac{80}{s} \text{ and } G(s) = \frac{K_c K_p}{\tau s + 1}$$

then

$$e_{ss} = \lim_{s \to o} \left(\frac{80}{1 + \frac{K_c K_p}{\tau s + 1}} \right) = \frac{80}{1 + K_c K_p}$$

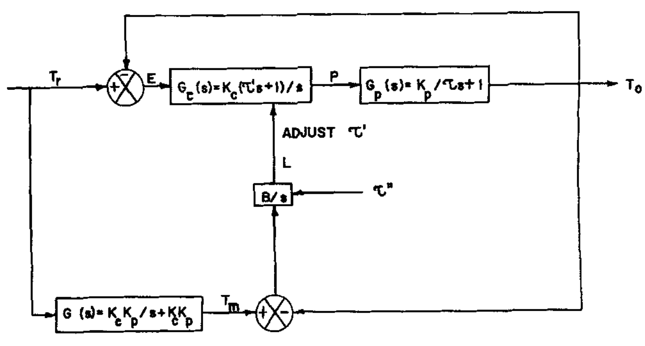


Fig. 2. Adaptive temperature control.

The higher the value of K_c, the less the steady state error. However, to accomplish this may require an excessively high value of K_c; also, with a high gain the existence of higher derivatives in the process may become critical. To eliminate the steady state error integral plus proportional control can be used. In this case the error is zero.

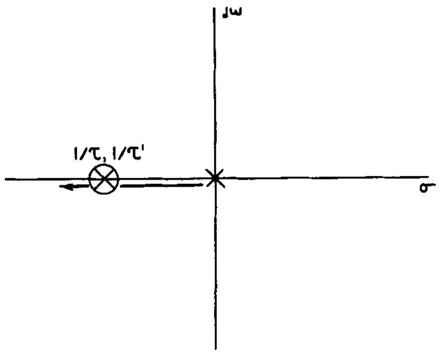
$$e_{ss} = \lim_{s \to 0} \left[\frac{80}{1 + \dfrac{K_c K_p \, (\tau' s + 1)}{s(\tau s + 1)}} \right] = 0$$

Now, unfortunately, the problem of overshoot is critical when $\tau >> \tau'$. As $\tau' \to \tau$ the overshoot goes to zero, so the ideal controller would have the adaptive feature of making $\tau' \to \tau$ during the transient response of the system. The block diagram for the simulation of this adaptive temperature control is given in Figure 2.

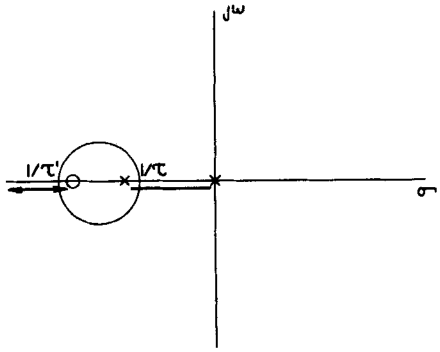
ANALOG COMPUTER RESULTS

Transient Response of Adaptive System

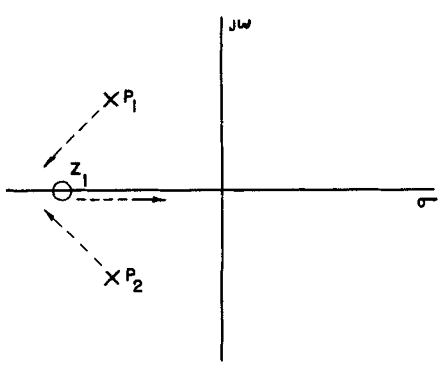
If $\tau = \tau'$, the root-locus diagram of the system without adaptive control is



and when $\tau \neq \tau'$



When the adaptive feature is operative, it has the effect of moving the zero toward the $j\omega$ axis, which in turn causes the poles to move to the left and toward the real axis as shown below.



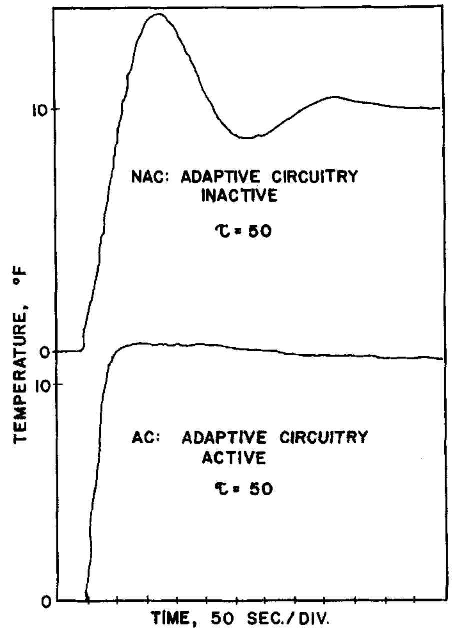


Fig. 3. System response to step input ($\tau = 50$).

Figure 3 provides an example of the effectiveness of the adaptive control. For τ equal to 5, 10, and 50 the percent overshoot was reduced from 6.5 to 0.9, 14.5 to 1.1, and 40.8 to 7.6, respectively. The corresponding settling times (defined as 2% of the final value) were reduced from 120 to 20, 160 to 20, and 620 to 400 sec., respectively.

When $K = K(t)$ and $B(s) = B/s$, we have

$$l_{ss} \to \frac{[K - K(t)]\, BT_\tau}{KK(t)}$$

When $K > K(t)$ we expect a larger correction signal and a steady state value greater than zero. If $\tau > \tau'$ this will improve the response while if $\tau < \tau'$ it will hinder the response and vice versa. The problem that exists is that with each succeeding transient $l(t)$ increases, eventually disrupting the system. If $K(t) > K_{\text{model}}$ the system will go unstable after several transients, since $l(t)$ has an increasingly larger negative value, causing the zero to move further away and the poles closer to the imaginary axis.

Thus far the adaptive circuitry is effective for positive step and ramp inputs. Consider now what happens when a negative step or a rectangular pulse is the input. When $\tau > \tau'$ the negative step causes $T_o(t) > T_{om}(t)$, resulting in a negative error signal. But this should occur only when $\tau < \tau'$; the result is a correction in the opposite direction with the poles of the system closer to the imaginary axis and a poorer transient response with eventual instability. One method of eliminating this difficulty would be to use the absolute value of the error between system and model responses for correction of τ'. This technique, however, would correct for $\Delta\tau$ either positive or negative but not both. By changing the sign of the error signal according to the sense of the input we avoid this problem. That is, when the derivative of the input is negative the error signal changes sign and vice versa. Figures 4 and 5 illustrate that this scheme will compensate for a time constant less than or greater than the design value τ''. The correction signal shown in Figure 6 indicates that after a series of pulses the system reaches the condition where $\tau' = \tau$, since $l(t)$ continues to increase, when the sign of the error changes, until it reaches the value of $\Delta\tau$ (or very close). At this point there is no error between the system and model and $l(t)$ remains constant. It is important to

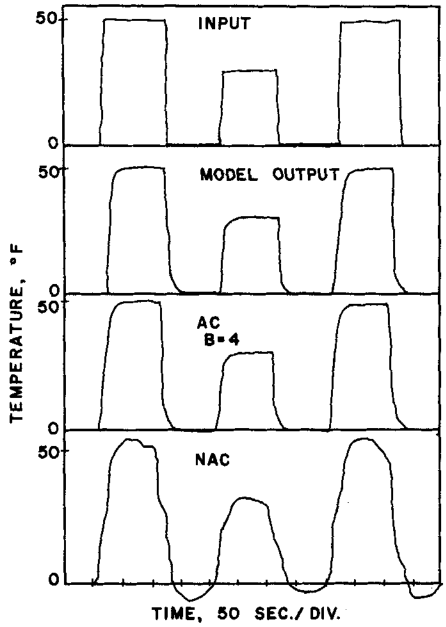


Fig. 4. System response to rectangular pulses ($\tau = 10$).

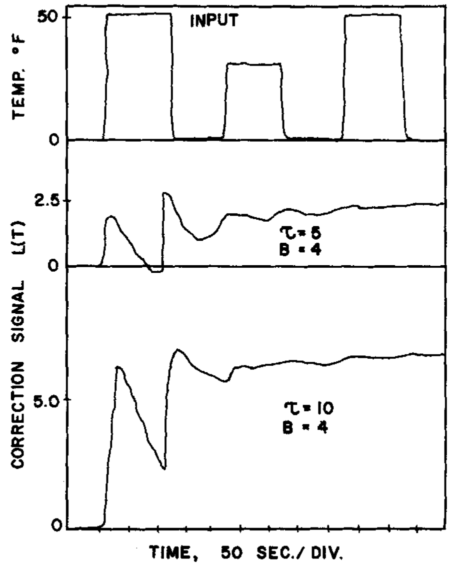


Fig. 6. Correction signal for rectangular pulses.

note that this is only true when the system and model gains are equal.

Consider now the choice of $B(s) = B$. Here the only operation performed on the error between system and model is one of multiplication. For the same value of B the overshoot is more effectively reduced when $B(s) = B/s$ is used. The initial correction when $B(s) = B$ is more rapid but it also decays more rapidly, which is expected, since the $\Delta\tau$ portion of $l(t)$ is the coefficient of a second-order derivative term. If $T_o(t)$ does not decay before the first crossover between $T_o(t)$ and $T_{om}(t)$, then $l(t)$ will be in the wrong direction with the result that the transient portion of $T_o(t)$ lasts even longer. With $B(s) = B/s$, $l(t)$ is slower to peak but also decays more slowly giving a correction signal that does not change sign.

Increasing the value of B has the effect of reducing the percent overshoot. In the case of a single step input it multiplies $l(t)$ during the transient, making $l(t)_{\max}$ closer to $\Delta\tau$. Too large a value of B may cause $l(t)_{\max}$ to be greater than $\Delta\tau$. Figure 7 represents the solution of Equation (10) for various values of τ and B, with each point corresponding to a solution for one particular τ and B, with each point corresponding to a solution for one particular τ and B, while each curve is a set of solutions at constant B. Since the magnitude of $l(t)$ is approximately given by BT_r, the size of the step input as well as the value of B affects the degree of adaptability; the larger the magnitude of the step the smaller the required B to maintain a given percent overshoot.

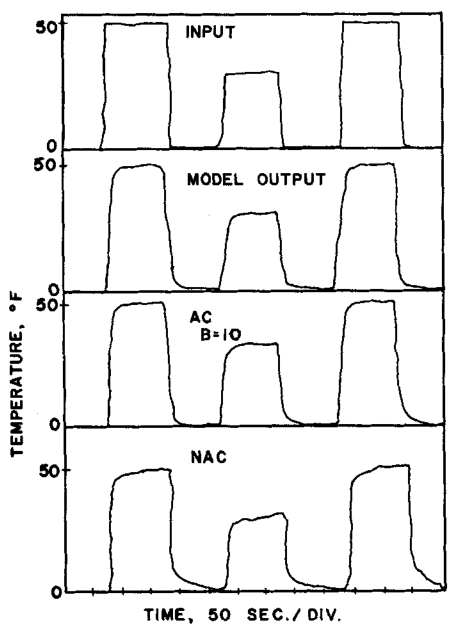


Fig. 5. System response to rectangular pulses ($\tau = 0.33$).

Effect of an Additional Real Pole

An important consideration is the effect on the control system if there is another undetected pole, which in this process may be associated with the valve transfer function. This pole might be further from the origin and might not have been detected in the dynamic analysis of the process. If $1/\tau_2$ indicates the position of the additional pole the process transfer function becomes

$$G_p(s) = \frac{K_p}{(\tau_2 s + 1)\ (\tau s + 1)}$$

and the system transfer function

$$G_o(s) = \frac{K(\tau' + 1)}{\tau_2 s^3 + (\tau_2 +) s^2 + (1 + K\tau')\, s + K}$$

If τ_2 had not been previously detected in the dynamic analysis of the process we can reasonably expect that $\tau_2 <<< \tau$. Therefore $\tau_2 \tau s^2 \approx 0$ and $\tau_2 + \tau \approx \tau$, so that as $1/\tau_2$ moves further from the origin its effect will become negligible.

Effect of a Pure Time Delay

A pure time delay may occur in the forward path or the feedback path. In the case of the delay in the forward path the result is similar in effect to the addition of an additional pole. As τ_d decreases its effect becomes negligible. When the delay appears in the feedback path the situation is quite different. If τ_1 is the first value, then at any time before the first crossing, $T_o(\tau)$ will be less than $T_o(\tau_1)$. If τ decreases, then $T_o(\tau)$ will be greater than $T_o(\tau_1)$. With a pure delay in the feedback path the error

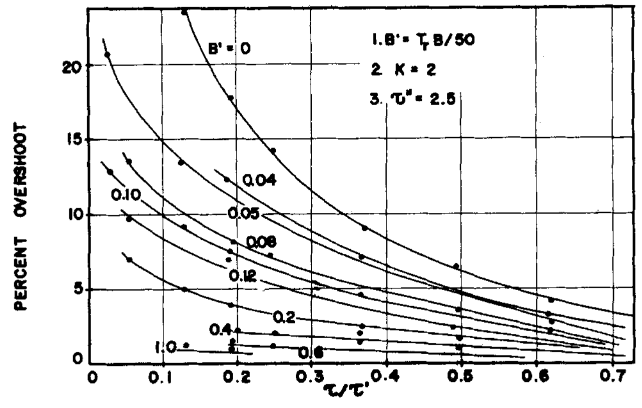


Fig. 7. Effect of B on percent overshoot.

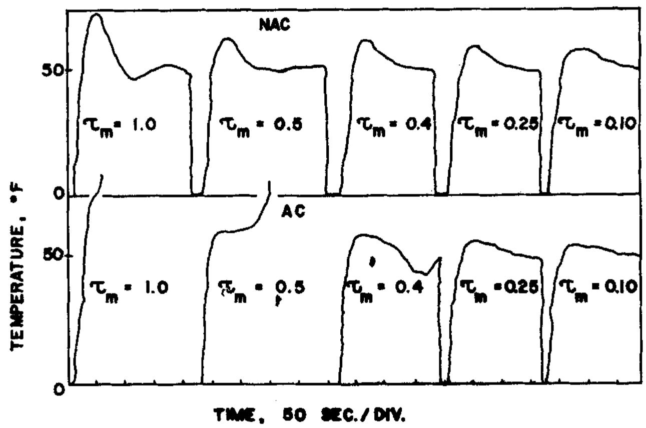


Fig. 8. System response, effect of measurement lag ($\tau = 10$, $B = 1$).

signal to the controller is larger than does actually exists, causing a response such that $T_o(t)$ before the first crossing is greater than the value of $T_o(t)$ when no delay is present. Therefore the response as the delay increases looks like the response when τ decreases even though τ may be greater than τ'. If the delay effect is greater than the $\Delta\tau$ effect, the adaptive circuitry corrects in the wrong direction with a resulting instability. As $\tau_d \rightarrow 0$ the undesirable effect becomes negligible. A pure delay in the model that is greater than or equal to the system delay would eliminate the problem.

If a delay $e^{-\tau_m s}$ is put in the model then the transfer function relating $L(s)$ to $T_r(s)$ becomes

$$\frac{L(s)}{T_r(s)} = \frac{B(s)K[s^2(\Delta\tau) + K\tau' s(e^{-\tau_d s} - e^{-\tau_m s}) + K(e^{-\tau_d s} - e^{-\tau_m s})]}{(s + Ke^{-\tau_m s})\,[\tau s^2 + (1 + Ke^{-\tau_d s})\,s + Ke^{-\tau_d s}]}$$

If $\tau_m = \tau_d$, then l_{ss} is the same as given by Equation (9). If $\tau_m \neq \tau_d$, all the terms in the numerator of Equation (28) are positive if $\tau_m > \tau_d$.

Effect of Measurement Lag

Another important consideration is the effect of a measurement lag, since in any practical process control scheme there would be some lag associated with the sensing device. As the time constant τ_m of the measurement lag increases the adaptive response approaches an unstable condition. This is illustrated in Figure 8. In two cases, $\tau_m = 1$ and $\tau_m = 0.5$, instability results as the correction signal increases negatively, resulting in a negative coefficient of the system differential equation. When a measurement lag is also put into the model the instability can be eliminated. It is not necessary for the time constant τ_{mm} of the model measurement lag to cancel exactly the one in the process.

CONCLUSIONS

1. The adaptive temperature control scheme for a chemical reactor reduces the overshoot to essentially zero for a single-step input when large differences exist between the actual process time constant and the original setting of the controller parameter.
2. Increase of the value of B decreases the percent overshoot during the first transient response and increases the rate of convergence of the adaptive circuitry over a series of transients.
3. By determining the sense of the input, the adaptive scheme provides excellent transient response to a series of rectangular pulses when the process time constant is either larger or smaller than the assumed design value.
4. Small differences between the process gain and the model gain cause no difficulty. Large differences will cause either an unstable or extremely overdamped response, depending on whether the model gain is less than or greater than the process gain.
5. A pure time delay in the forward path has no effect on the stability of the adaptive system.
6. As the time of a pure delay in the feedback path increases, the adaptive system becomes unstable.
7. When the time constant associated with a process measurement lag increases the adaptive system response becomes unstable. The instability is eliminated by having a measurement lag in the model with a time constant approximately equal to that of the process.
8. The criteria for maintaining system stability can be determined by the use of Lyapunov functions.
9. The use of linear analysis of the system differential equation when the correction signal $l(t)$ is not introduced into the system provides a means of designing the adaptive circuitry and approximating the effect when $l(t)$ is introduced into the controlled process.

ACKNOWLEDGMENT

The support of the National Science Foundation through a faculty fellowship is gratefully acknowledged.

NOTATION

A = area
C_a = concentration in reactor
C_{af} = concentration of product
C_{ai} = concentration of feed
C_p = heat capacity
k = specific rate constant
Q = heat of reaction
T = temperature in reactor
T_f = temperature of product
T_i = temperature of feed
T_j = jacket temperature
U = overall heat transfer coefficient
v = feed rate
V_T = volume
ρ = density

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Manuscript received April 15, 1966; revision received September 2, 1966; paper accepted September 3, 1966.

Friction Factors and Pressure Drop for Sinusoidal Laminar Flow of Water and Blood in Rigid Tubes

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From the Navier-Stokes equations and a modified Fanning equation, a theoretical equation was derived for computing friction factors and pressure drop for sinusoidal flow in rigid pipes. The friction factor equation was $f = (\pi/16S)(16/N_{Re})$, which is analogous to the usual laminar flow equation. The factor S is dependent on the frequency and kinematic viscosity and is easily computed. Friction factors were calculated from experimental data and it was found that the theoretical friction factors predicted the experimental values to within less than 5%.

Pulsatile flow has received an increased amount of attention from engineers and physiologists in recent years, who recognize that the pulsatile flow phenomenon exists in pumping systems, heat and mass transfer operations as well as in the circulatory blood flow circuit in living organisms.

Chantry et al. (*3*) applied pulsation to liquid-liquid extraction and De Maria and Benenati (*5*) studied the effect of pulsation in batch thermal diffusion. Krasuk and Smith (*9*) and Shirotsuka (*17, 18*) studied mass transfer in a pulsed column and Linford (*13*) verified that the superimposed pulsation did not affect the streamline character of flow. Many investigators reported that the pulsation increased the efficiency of the mass and heat transfer processes.

Theoretical studies of pulsatile flow in a circular tube have been done by Sexl (*16*), Uchida (*22*), Womersley (*25*), Lambossy (*11*), and Kusama (*10*). Womersley and others derived the equation for average flow rate starting with the Navier-Stokes equation and a sinusoidal pressure gradient $dp/dz = A\ e^{i\omega t}$.

Physiologists have been interested in pulsatile flow because of its applicability to the circulatory system. Landowne (*12*) and Taylor (*21*) studied the propagation of a pulse wave in arteries and Bergel (*1*) investigated the dynamic elastic properties of the wall during pulsatile flow. McDonald (*15*), Evans (*6*), and Caro (*2*) established a relationship between pressure and flow in arteries by using Womersley's theory (*23, 24*).

To determine the power requirement for designing a pulsatile flow system, it is important to find the energy loss expressed as the friction at the tube wall. Shirotsuka (*17, 18*) studied mass, heat, and momentum transfer in pulsatile flow. He proposed to represent the fractional increments of pulsatile friction factor vs. steady flow values by the nondimensional empirical equation

$$\frac{f_p - f}{f} = 2.5 \times 10^3 \left(\frac{A\,n}{u}\right)^{1.6} \left(\frac{A\,n\,D}{\alpha}\right)^{-0.6} \left(\frac{D}{A\,n}\right)^{0.5}$$

Kusama (*10*) determined the time-averaged friction factor f_p for pulsatile flow by finding the work necessary to overcome the friction force over the period T of a pulse.

All of the preceding derivations involved extensive calculations and also required steady flow values to determine the friction factor for pulsatile flow. The work of Hershey and Song (*20*) was undertaken to develop a theoretical equation for pulsatile flow which would be analogous to the steady flow laminar friction factor equation

$$f = 16/N_{Re}$$

DERIVATION OF A FRICTION FACTOR EQUATION FOR SINUSOIDAL LAMINAR FLOW

If the upstream pressure is $P_0\ (1 + \sin \omega t)$, then for an incompressible fluid in a rigid tube, the downstream pressure may be expressed by a modified Fanning equation:

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